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Percolation on a Bethe lattice with multi-neighbour bonds—exact results

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Abstract. Percolation problems on a Bethe lattice with Lth-nearest-neighbour bonds are treated exactly by a generalised recursive method. For the site percolation, both the cases without and with interbranch bonds are considered. Formal expressions for the critical percolation p_c, percolation probability $P(p)$ (near p_c) and the mean cluster size $S(p)$ ($p < p_c$) are obtained for any K ($K+1$ = degree of the Bethe lattice) and L. For the bond percolation, only the case of $K=2$ and $L=2$ is considered. The method described here can be extended to other more complicated branching media including decorated Bethe lattices.

1. Introduction

The percolation problem (for reviews see Stauffer 1979, Essam 1980) has attracted much attention in recent years because of its close relationship with the thermal critical phenomena (Kasteleyn and Fortuin 1969). Up to now, there are only a few cases where exact results have been found. Percolation on a Bethe lattice is the simplest branching medium which can be solved exactly. Fisher and Essam (1961) were the first to solve random percolation problems on Bethe or decorated Bethe lattices. Recently, more complicated percolation problems on a Bethe lattice with lattice anisotropy or correlations have been proposed and solved (Turban 1979a, b, Turban and Guilmin 1979, Chalupa *et al* 1979). However, all those systems studied are restricted to nearest-neighbour bonds only. For percolation systems with bonds connecting Lth nearest neighbours, only the one-dimensional case has been solved and non-universal critical behaviour is found (Klein *et al* 1978, Zhang and Shen 1982, Zhang *et al* 1983, Li *et al* 1983).

In this work the recursive method which is widely used in the Bethe lattice is extended to treat the case of bonds connecting Lth nearest neighbours. For the site percolation, we consider in §§ 2 and 3 respectively the cases without and with interbranch bonds. Bethe lattices without and with interbranch bonds are shown in figures 1(a) and (b) respectively for the case $K=2$ and $L=2$, where $K+1$ is the degree of a Bethe lattice. A Bethe lattice without interbranch bonds is rather unusual. From figure 1(a) one can see that the first and second generations are different from the rest. For any finite L, we will have the first L generations which are different from the rest and also differ among themselves. Strictly speaking, a Bethe lattice without interbranch bonds is not a regular pseudo-lattice. The reasons why we study such a case are two-fold. Firstly, it is the simplest case to consider for the general value of K and L. It also makes the generalised recursive method easy to present. Secondly, as we will see in § 2, the critical behaviour (e.g. percolation threshold and critical

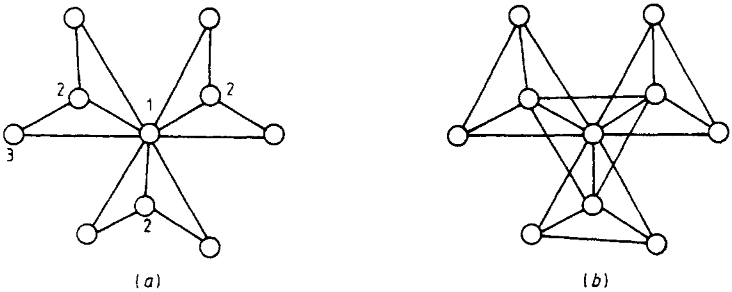

Figure 1. A Bethe lattice with $K=2$ and $L=2$. (a) No interbranch bonds; (b) with interbranch bonds.

exponents) which is determined by the recursion relations depends not on the first L generations but on the rest of the lattice. By studying both cases without and with interbranch bonds, one can see the effects of interbranch bonds on the critical behaviour, particularly the critical percolations. A Bethe lattice without interbranch bonds has been used recently in studying the modulated phase of an Ising system with competing nearest-neighbour and next-nearest-neighbour interactions (Vannimenus 1981). Section 4 is devoted to the bond percolation problem with interbranch bonds. Here, only the case with $K=2$ and $L=2$ is considered. It is straightforward to apply the same method to the case of bond percolation on a Bethe lattice without interbranch bonds. Since this is not a regular Bethe lattice and is of less interest, we will not consider this case here. Finally a summary is given in § 5.

2. Site percolation without interbranch bonds

In this section we consider the simplest case of site percolation on a Bethe lattice without interbranch bonds. Considering the case of $K=2$ and $L=2$ (figure 1(a)), we define Q_0 (figure 2) as the probability that the number of occupied sites in a single branch emerging from sites 1 and 2 is finite when site 2 is occupied. Similarly, Q_1 is defined as the probability that the number of occupied sites in a single branch emerging from sites 1 and 2 is finite when site 1 is occupied and site 2 is empty. Since $L=2$, the configurations of a chain of two sites 1–2 have to be considered to determine the connectivity property of the branch emerging from this chain. In the site percolation, if a site on the chain is occupied, then all the other sites preceding that site become irrelevant. For $L=2$, there are only two independent variables, Q_0 and Q_1. It is easy

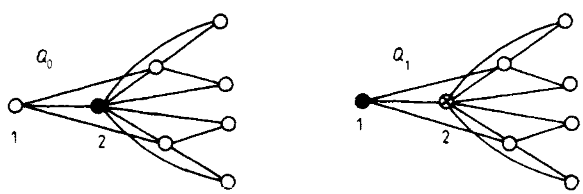

Figure 2. Illustrations of the definitions of Q_0 and Q_1 for $L=2$. ● denotes occupied site and ⊗ denotes empty site.

to obtain the recursion relations

$$Q_0 = (qQ_1 + pQ_0)^2 \qquad Q_1 = (q + pQ_0)^2 \tag{2.1}$$

where p is the site occupation probability and $q = 1 - p$. The above recursive method is easily generalised to the case of any L and K. In this case, there are L independent variables. We define Q_i ($i = 0, 1, \ldots, L-1$) as the probability that the number of occupied sites emerging from a single branch 1–2–3–. . .–L is finite when the (L–i)th site is occupied while the sites from L–$i+1$ to L are empty. The recursion relations then become

$$Q_i = (qQ_{i+1} + pQ_0)^K \qquad i = 0, 1, \ldots, L-1$$

with

$$Q_L = 1. \tag{2.2}$$

Now consider an occupied site. This belongs to a finite cluster with probability $(1 - P(p))$, when $K+1$ branches emerging from it are finite. So we have

$$1 - P(p) = (qQ_1 + pQ_0)^{K+1} \tag{2.3}$$

where $P(p)$ is usually called the percolation probability. Equation (2.2) always has a trivial solution $Q_i \equiv 1$ for all i from 0 to $L-1$. As we increase p from 0 up to a critical point p_c, a second non-trivial solution appears and Q_i starts to decrease from 1. This indicates the existence of an infinite cluster. Let λ $(= p - p_c = q_c - q)$ be a small parameter. Both sides of (2.2) can be expanded in terms of λ. To first order in λ, we find

$$\sum_{j=0}^{L-1} A_{ij} Q_j'(\lambda = 0) = 0 \qquad i = 0, 1, \ldots, L-1$$

with

$$Q_L'(0) = 0$$

$$A_{ij} = K(q_c\delta_{i+1,j} + p_c\delta_{0,j}) - \delta_{ij} \tag{2.4}$$

where Q_j' $(\lambda = 0)$ is the derivative of Q_j with respect to p evaluated at p_c $(\lambda = 0)$. The critical point p_c is determined by the condition for the existence of a non-trivial solution of $Q_j'(0)$. From (2.4) the condition $\det(A_{ij}) = 0$ gives

$$Kp_c[1 + Kq_c + (Kq_c)^2 + \ldots + (Kq_c)^{L-1}] = 1 \tag{2.5}$$

which determines p_c for any K and L. It can be proved rigorously that (2.5) gives one, and only one, real root q_c in the range (0, 1) and $q_c(K, L)$ approaches one as K or L is increased.

The percolation probability $P(p)$ near p_c can be found by expanding (2.3) to first order in λ. After simple manipulations, we find

$$P(p) = -[(K+1)/K]Q_0'(0)(p - p_c) + \mathrm{O}[(p - p_c)^2] \tag{2.6}$$

where $Q_0'(0)$ can be obtained analytically and is expressed as (see the appendix)

$$Q_0'(0) = \frac{-2K^2(x-1)[x^{L+1} - L(K-1)]}{x^{L+1}(x-1)^2 - (K-1)^2 x[x^L - (2L+1)x + 2(LK-1)]} \tag{2.7}$$

where $x = Kq_c$. Away from p_c, $P(p)$ has to be determined numerically by solving (2.2). We have solved (2.2) numerically for the cases of $K = 2$–4 and $L = 2$–5. The values for $p_c(K, L)$ are shown in figure 3 (full lines). The percolation probability $P(p)$ for some values of K and L is shown in figure 4 (full curves).

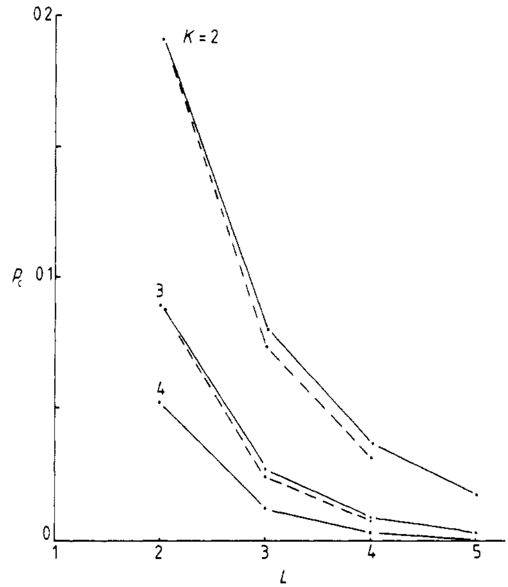

Figure 3. p_c for various values of K and L. The full lines denote no interbranch bonds and broken lines denote interbranch bonds.

In order to find the mean cluster size $S(p)$, we use the 'ghost site' method (Turban 1979a, b). Here we give a brief review of the method. Let $P_s(p)$ be the probability that a given site belongs to a finite cluster of s sites. From the definitions of $P(p)$ and $P_s(p)$, we have the following sum rule

$$(1-P(p))=\sum_s{}' P_s(p)/p \tag{2.8}$$

where the prime on the sum indicates the exclusion of the infinite cluster. The mean

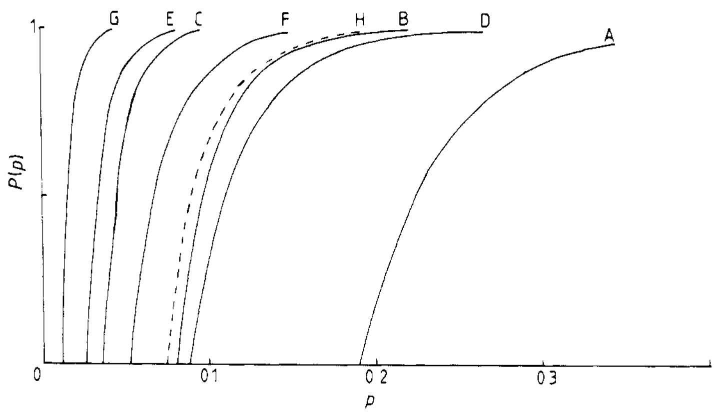

Figure 4. Percolation probability $P(p)$ for various values of K and L which are denoted by (K, L). The full curves denote no interbranch bonds and the broken curves denote interbranch bonds. The values of (K, L) for the curves are as follows: A, (2, 2); B, (2, 3); C, (2, 4); D, (3, 2); E, (3, 3); F, (4, 2); G, (4, 3); H, (2, 3).

cluster size $S(p)$ is defined as

$$S(p) = \sum_s{}' sP_s(p) / \sum_s{}' P_s(p). \tag{2.9}$$

In the presence of the 'ghost site', the probability that a site belongs to a finite cluster of size s becomes

$$P_s(p, h) = P_s(p)(1-h)^s \tag{2.10}$$

where h is the 'ghost field' which connects the 'ghost site' to every occupied site. The percolation probability $P(p)$ is also h dependent and (2.8) becomes

$$(1 - P(p, h)) = \sum_s{}' P_s(p, h)/p. \tag{2.11}$$

From (2.8)–(2.11), it is easy to see that $S(p)$ is given by

$$S(p) = (\partial P(p, h)/\partial h)|_{h=0}/(1 - P(p)). \tag{2.12}$$

We now come back to the Bethe lattice. In the presence of the 'ghost site', (2.2) and (2.3) become respectively

$$Q_i(p, h) = [qQ_{i+1}(p, h) + p(1-h)Q_0(p, h)]^K \qquad i = 0, 1, \ldots, L-1$$

with

$$Q_L(p, h) \equiv 1 \tag{2.13}$$

and

$$(1 - P(p, h)) = (1-h)[qQ_1(p, h) + p(1-h)Q_0(p, h)]^{K+1}. \tag{2.14}$$

Since the occupied site chosen in defining $(1 - P(p))$ of (2.13) has to be disconnected from the 'ghost site', this gives a factor $(1-h)$ in front of the bracket on the right-hand side of (2.14). Taking the partial derivatives with respect to h on both sides of (2.14) and evaluating at $h = 0$, we find using (2.12)

$$S(p) = 1 - \frac{(K+1)Q_0(p)}{1 - P(p)}\left[q\left(\frac{\partial Q_1}{\partial h}\right)_{h=0} + p\left(\frac{\partial Q_0}{\partial h}\right)_{h=0} - pQ_0(p)\right] \tag{2.15}$$

where (2.2) and (2.3) have been used to derive (2.15). Differentiating both sides of (2.13) with respect to h and evaluating at $h = 0$, we find a set of inhomogeneous equations

$$\left(\frac{\partial Q_i}{\partial h}\right)_{h=0} = \frac{KQ_i(p)}{qQ_{i+1}(p) + pQ_0(p)}\left[q\left(\frac{\partial Q_{i+1}}{\partial h}\right)_{h=0} + p\left(\frac{\partial Q_0}{\partial h}\right)_{h=0} - pQ_0(p)\right]$$

$$i = 0, 1, \ldots, L-1 \tag{2.16}$$

with

$$(\partial Q_L/\partial h)_{h=0} = 0$$

where (2.2) has been used to derive (2.16). Using (2.16) when $i = 0$, (2.15) can be rewritten as

$$S(p) = 1 - \frac{(K+1)(\partial Q_0/\partial h)_{h=0}}{K(1 - P(p))}(qQ_1(p) + pQ_0(p)). \tag{2.17}$$

For $p > p_c$, we have to solve (2.16) numerically for $(\partial Q_0/\partial h)_{h=0}$ in order to find $S(p)$.

However, when $p<p_c$ we have $Q_i(p)=1$ for all i and $P(p)=0$. Equation (2.16) is then simplified to

$$\sum_{j=0}^{L-1} A_{ij}(\partial Q_j/\partial h)_{h=0}=Kp \qquad i=0,1,\ldots,L-1 \tag{2.18}$$

with

$$(\partial Q_L/\partial h)_{h=0}\equiv 0$$

where A_{ij} is given by (2.4). (2.18) is easy to solve for $(\partial Q_0/\partial h)_{h=0}$. We obtain finally the following formal expression of $S(p)$ for any K and L:

$$S(p)=\frac{1+p[1+Kq+(Kq)^2+\ldots+(Kq)^{L-1}]}{1-Kp[1+Kq+(Kq)^2+\ldots+(Kq)^{L-1}]} \qquad p<p_c. \tag{2.19}$$

$S(p)$ diverges when the denominator of (2.19) becomes zero. This gives exactly (2.5) for determining the critical point. In one dimension, putting $K=1$, (2.19) becomes

$$S(p)=(2-q^L)/q^L. \tag{2.20}$$

This is exactly the result obtained by the generating function method (Klein *et al* 1978). It is easy to see that (2.6), (2.7) and (2.19) give the critical exponents $\beta=1$ and $\gamma=1$ for all $K>1$ and L. The universality is expected to hold for dimensionality greater than one if higher-neighbour bonds are taken into account. This has been demonstrated by Monte Carlo simulations on common lattices (Hoshen *et al* 1978, 1979).

3. Site percolation with interbranch bonds

If interbranch bonds are considered, recursion relations (2.2) have to be modified when $L\geqslant 3$. For instance, when $L=4$, the recursion relations are

$$Q_0=(qQ_1+pQ_0)^K \tag{3.1}$$

$$Q_1=(qQ_2+pQ_0)^K \tag{3.2}$$

$$\begin{aligned}Q_2&=(qQ_3)^K+\binom{K}{1}(qQ_2)^{K-1}(pQ_0)+\binom{K}{2}(qQ_2)^{K-2}(pQ_0)^2\\&\quad+\ldots+\binom{K}{K-1}(qQ_2)(pQ_0)^{K-1}+(pQ_0)^K\\&=q^K(Q_3^K-Q_2^K)+Q_1\end{aligned} \tag{3.3}$$

$$\begin{aligned}Q_3&=q^K+\binom{K}{1}(qQ_2)^{K-1}(pQ_0)+\ldots+\binom{K}{K-1}(qQ_2)(pQ_0)^{K-1}+(pQ_0)^K\\&=q^K(1-Q_2^K)+Q_1.\end{aligned} \tag{3.4}$$

Figure 5 shows Q_i $(i=0,1,2,3)$ for the case of $K=3$. Considering Q_2, if none of the sites 5, 6 and 7 emerging from the branch 1–2–3–4 are occupied, the probability for this branch to be a finite cluster is $(qQ_3)^K$. However, if one of the sites 5, 6, 7 (say site 5) is occupied, then the probability for the sub-branch 1–2–3–4–5 to be a finite cluster is pQ_0, while for the other two sub-branches the probability is $(qQ_2)^2$.

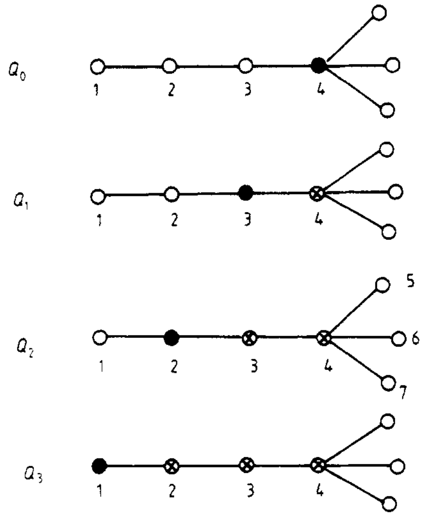

Figure 5. Illustrations of the definition of Q_i, $i = 0$–3, for $K = 3$ and $L = 4$ (with interbranch bonds). Only the nearest-neighbour bonds are drawn. All the other bonds from $L = 2$–4 which are not shown here do exist.

This is because site 5 is closer to the other two sub-branches than site 2. In the case of site percolation, site 2 becomes irrelevant. Since there are three ways to occupy a single site among sites 5, 6 and 7, we get a term $\binom{3}{1}$ $(qQ_2)^{3-1}(pQ_0)$, which is the second term of (3.3). Similarly other terms in (3.3) can be generated. For any L, the recursion relations are

$$\begin{aligned} Q_i &= (qQ_{i+1} + pQ_0)^K \qquad i = 0, 1 \\ Q_i &= q^K(Q_{i+1}^K - Q_2^K) + Q_1 \qquad i = 2, 3, \ldots, L-1 \end{aligned} \tag{3.5}$$

with

$$Q_L \equiv 1.$$

Using the same method as described in § 2, we expand Q_i in the vicinity of p_c, with $\lambda = p - p_c$ as a small parameter, to first order in λ; equations (3.5) give

$$\sum_{j=0}^{L-1} B_{ij} Q_j'(\lambda = 0) = 0 \qquad i = 0, 1, \ldots, L-1 \tag{3.6}$$

with

$$Q_L'(0) = 0$$

where

$$B_{ij} = \begin{cases} x\delta_{i+1,j} + (K-x)\delta_{0,j} - \delta_{i,j} & i = 0, 1 \\ y\delta_{i+1,j} + \delta_{1,j} - y\delta_{2,j} - \delta_{i,j} & i = 2, 3, \ldots, L-1 \end{cases} \tag{3.7}$$

with $x = Kq$ and $y = Kq^K$. The critical point p_c is determined by the condition for the existence of a non-trivial solution of (3.6). From $\det(B_{ij}) = 0$, we find, with some manipulation, the equation for q_c as

$$[x_c^2 + (1-K)x_c]y_c^{L-2} + (1-K)(1 + y_c + y_c^2 + \ldots + y_c^{L-2}) = 0 \tag{3.8}$$

where $x_c = Kq_c$, $y_c = Kq_c^K$. It can also be proved that (3.8) has a unique real root in (0, 1), and q_c increases to one as K or L is increased. Equation (3.8) is solved numerically for $K = 2, 3$ and $L = 3, 4$. The values of p_c are shown in figure 3 (broken lines). As was expected, the existence of interbranch bonds gives lower values of p_c for all $K > 1$ and $L \geqslant 3$. The percolation probability is again given by (2.3). Equation (3.5) is solved numerically for the case $L = 3$ and $K = 2$. $P(p)$ is plotted in figure 4 (broken curve) for comparison. Near p_c the analytic expression of $P(p)$ for any K and L can be obtained.

To second order in λ (3.5) gives

$$\sum_{j=0}^{L-1} B_{ij} Q_j''(0) = C_i \qquad i = 0, 1, \ldots, L-1 \tag{3.9}$$

where

$$C_i = \begin{cases} [(1-K)/K](Q_i'(0))^2 + 2K(Q_{i+1}'(0) - Q_0'(0)) & i = 0, 1 \\ 2K^2(y/x)(Q_{i+1}'(0) - Q_i'(0)) - (K-1)y[(Q_{i+1}'(0))^2 - (Q_2'(0))^2] & \\ & i = 2, 3, \ldots, L-1. \end{cases} \tag{3.10}$$

B_{ij} of (3.9) is given by (3.7). The solvability condition of (3.9) determines $Q_i'(0)$. After some lengthy but straightforward manipulations, parallel to the one shown in the appendix, we find

$$\begin{aligned} Q_0'(0) = {} & 2K^2x[x(2x-K+1)y^{L-2} + KyB(K,L)] \\ & \times (x^2y^{L-2}[(2K-3)x^2 - 3(K-1)^2x + (K-1)^3] \\ & + K(K-1)y\{2[x^2 + (1-K)x + 1 - K]B(K,L) - C(K,L)\})^{-1} \end{aligned} \tag{3.11}$$

where

$$B(K,L) = [(K-1)/y(y-1)^2][y^{L-1} - (L-1)y + L - 2] \tag{3.12}$$

and

$$\begin{aligned} C(K,L) = {} & (y-1)^{-1}[x^2 + (1-K)x + 1 - K]^2 y^{L-3}(y^{L-2} - 1) \\ & + (y-1)^{-2}\{2(1-K)[x^2 + (1-K)x + 1 - K]y^{L-3}[y^{L-2} - (L-2)y + L - 3]\} \\ & + (1-K)^2(y-1)^{-3}[y^{2L-5} + (5-2L)y^{L-2} - (5-2L)y^{L-3} - 1]. \end{aligned} \tag{3.13}$$

The formal expression for the percolation probability $P(p)$ near p_c is obtained by substituting (3.11), (3.12) and (3.13) into (2.6).

In order to find the mean cluster size $S(p)$, we again use the 'ghost site' method. In the presence of the 'ghost site' the recursion relations (3.5) become

$$Q_i(p,h) = \begin{cases} [qQ_{i+1}(p,h) + p(1-h)Q_0(p,h)]^K & i = 0, 1 \\ q^K(Q_{i+1}^K(p,h) - Q_2^K(p,h)) + Q_1(p,h) & i = 2, 3, \ldots, L-1. \end{cases} \tag{3.14}$$

The percolation probability and the mean cluster size $S(p)$ are still given by (2.14) and (2.15) respectively. Taking the partial derivatives with respect to h on both sides of (3.14), we find

$$\left(\frac{\partial Q_i}{\partial h}\right)_{h=0} = \begin{cases} \dfrac{KQ_i(p)[q(\partial Q_{i+1}/\partial h)_{h=0} + p(\partial Q_0/\partial h)_{h=0} - pQ_0(p)]}{qQ_{i+1}(p) + pQ_0(p)} & i = 0, 1 \\ Kq^K[Q_{i+1}^{K-1}(p)(\partial Q_{i+1}/\partial h)_{h=0} - Q_2^{K-1}(p)(\partial Q_2/\partial h)_{h=0}] + (\partial Q_1/\partial h)_{h=0} & \\ & i = 2, 3, \ldots, L-1 \end{cases} \tag{3.15}$$

with

$$(\partial Q_L/\partial h)_{h=0} = 0.$$

Since the expressions for $(\partial Q_i/\partial h)_{h=0}$ are the same as (2.16) for $i = 0$ and 1, (2.17) is still a valid expression for the mean cluster size $S(p)$. Here, $(\partial Q_0/\partial h)_{h=0}$ is determined by (3.15), whereas Q_i and $P(p)$ are determined by (3.5) and (2.3). For $p > p_c$ the above quantities have to be solved numerically. For $p < p_c$ we have $Q_i = 1$, $P(p) = 0$ and (3.15) can be written as

$$\sum_{j=1}^{L-1} B_{ij}\left(\frac{\partial Q_i}{\partial h}\right)_{h=0} = \begin{cases} Kp & i = 0, 1 \\ 0 & i = 2, 3, \ldots, L-1 \end{cases} \tag{3.16}$$

where B_{ij} is given by (3.7). Equation (3.16) is simple enough to solve for $(\partial Q_0/\partial h)_{h=0}$. Using (2.17), we find finally the following expression of $S(p)$ for any L and K:

$$S(p) = \frac{(1+p+px)y^{L-2} + (1+p-x)(1+y+y^2+\ldots+y^{L-3})}{[x^2+(1-K)x]y^{L-2} + (1-K)[1+y+y^2+\cdots+y^{L-2}]} \tag{3.17}$$

where $x = Kq$ and $y = Kq^K$. When $K = 1$, (3.17) reduces again to the known one-dimensional result (2.20). From the formal expressions of $P(p)$ near p_c ((3.11), (3.12), (3.13) and (2.6)) and $S(p)$ ($p < p_c$) (3.17), we find the critical exponents remain unchanged; $\beta = 1$ and $\gamma = 1$. This too is expected from the universality concept.

4. Bond percolation with interbranch bonds

In the bond percolation, there are no formal expressions which can include all values of K and L. Here we will only consider the simplest case of $K = 2$ and $L = 2$ (figure 6). The method described here can be easily generalised to treat higher values of K and L.

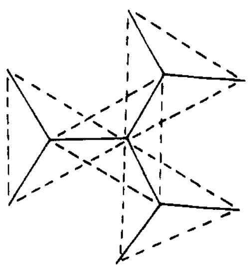

Figure 6. Bond percolation. A Bethe lattice with interbranch bonds for the case $K = 2$ and $L = 2$.

For a Bethe lattice with Lth-nearest-neighbour bonds, unlike site percolation, there are $(2^L - 1)$ independent variables for each configuration of a chain of L sites. For $K = 2$ and $L = 2$, Q_i ($i = 1, 2, 3$) are defined below (figure 7). Q_1 is the probability that the branch emerging from the chain 1–2 belongs to a finite cluster when both sites 1 and 2 are connected to the origin. Similar definitions are used for Q_2 and Q_3. We write $\phi_1^{(00)}$ as the probability that both sites 3 and 4 are connected to the origin when sites 1 and 2 are connected to the origin. Similar definitions are used for $\phi_1^{(0x)}$ and $\phi_1^{(xx)}$. For instance, $\phi_1^{(0x)}$ is the probability that one of the sites 3 or 4 is connected

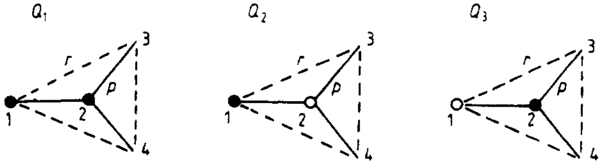

Figure 7. Illustrations of the definitions of Q_1, Q_2 and Q_3 for $K=2$ and $L=2$. ● denotes a site connected to the origin and ○ denotes a site disconnected to the origin.

to the origin when both sites 1 and 2 are connected to the origin. The recursion relation for Q_1 then becomes

$$Q_1=\phi_1^{(00)}Q_1^2+2\phi_1^{(0x)}Q_1Q_2+\phi_1^{(xx)}Q_2^2. \tag{4.1}$$

If p and r are respectively the bond occupation probability for the nearest- and next-nearest-neighbour bonds, using the exclusion–inclusion principle, it is easy to obtain

$$\phi_1^{(00)}=r(1-q^2s^2)+s(1-qs)^2 \qquad \phi_1^{(0x)}=qs^2(1-qs) \qquad \phi_1^{(xx)}=q^2s^2 \tag{4.2}$$

where $q=1-p$ and $s=1-r$. From (4.2), we find that $\phi_1^{(00)}+2\phi_1^{(0x)}+\phi_1^{(xx)}=1$. This is because the sum of the probability for various configurations of sites 3 and 4 must add up to one. The recursion relations for Q_2 and Q_3 are

$$\begin{aligned} Q_2&=\phi_2^{(00)}Q_3^2+2\phi_2^{(0x)}Q_3+\phi_2^{(xx)} \\ Q_3&=\phi_3^{(00)}Q_1^2+2\phi_3^{(0x)}Q_1Q_2+\phi_3^{(xx)}Q_2^2 \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} &\phi_2^{(00)}=r(1-s^2)+sr^2 &&\phi_3^{(00)}=r(1-q^2)+sp^2 \\ &\phi_2^{(0x)}=s^2r &&\phi_3^{(0x)}=spq \\ &\phi_2^{(xx)}=s^2 &&\phi_3^{(xx)}=q^2. \end{aligned} \tag{4.4}$$

The probability normalisation condition $\phi_i^{(00)}+2\phi_i^{(0x)}+\phi_i^{(xx)}=1$ again holds for $i=2$ and 3. Equations (4.1) and (4.3) possess a trivial solution $Q_i\equiv 1$ for all i. However, when the critical line $f(p_c, r_c)=0$ is reached from below, Q_i starts to decrease from 1. For some point (p, r) above but near the critical line, we write $f(p, r)=\varepsilon$ and $Q_i(p, r)=1-\eta_i$. Expanding (4.1) and (4.3) to first order in ε and η_i and using the probability normalisation conditions, we find

$$\eta_i=\sum_{j=1}^{3}a_{ij}\eta_j \qquad i=1,2,3 \tag{4.5}$$

where

$$\begin{aligned} &a_{11}=2[\phi_1^{(00)}+\phi_1^{(0x)}] &&a_{12}=2[\phi_1^{(0x)}+\phi_1^{(xx)}] \\ &a_{23}=2[\phi_2^{(00)}+\phi_2^{(0x)}] &&a_{31}=2[\phi_3^{(00)}+\phi_3^{(0x)}] \\ &a_{32}=2[\phi_3^{(0x)}+\phi_3^{(xx)}] &&a_{13}=a_{21}=a_{22}=a_{33}=0. \end{aligned} \tag{4.6}$$

From the condition $\det(a_{ij}-\delta_{ij})=0$, we find the equation for the critical line $f(p_c, r_c)=0$ as

$$\begin{aligned} &2q_cs_c^2(1+q_c-q_cs_c)-4(1-s_c^2-s_c^2r_c) \\ &\qquad\times[2q_cs_c^2(1+q_c-q_cs_c)-(s_cp_cq_c+q_c^2)]-1=0. \end{aligned} \tag{4.7}$$

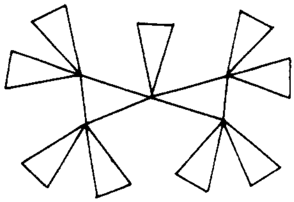

Figure 8. A triangular cactus of degree three.

When $r_c = 0$ ($s_c = 1$), (4.6) reduces to $q_c = \frac{1}{2}$, which is the result for the Bethe lattice ($K = 2$) with nearest-neighbour bonds only. When $p_c = 0$ ($q_c = 1$), (4.7) gives the known critical percolation of a triangular cactus of degree three (figure 8) as $4s_c^2(1 + r_c) = 3$ (Essam 1972). Unlike the case of site percolation where only one physical root of q_c exists in (2.5) and (3.8), (4.7) possesses another solution at $p_c = 0$; $2s_c^2(1 + r_c) = 1$. This solution gives a higher value of r_c and does not represent a critical point. Since the critical point is determined by the appearance of the first non-trivial solution of (4.1) and (4.3) with $Q_i < 1$ when p and/or r is increased from zero, only the solution with the lowest value of r_c represents a critical point. Equation (4.7) is solved numerically and the critical line is shown in figure 9.

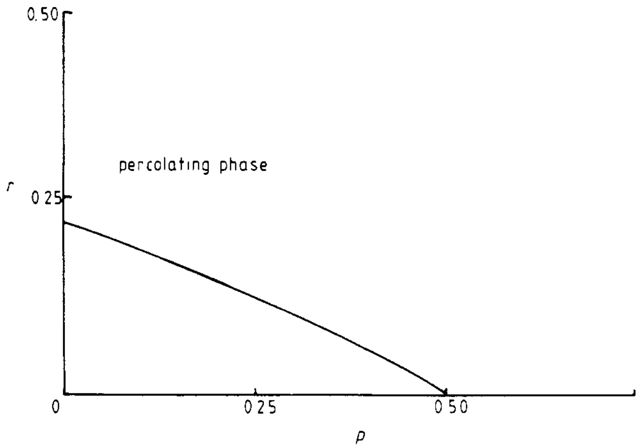

Figure 9. The critical line for bond percolation with $K = 2$ and $L = 2$ with interbranch bonds considered.

The bond percolation probability $P(p, r)$ is given by

$$1 - P(p, r) = aQ_1^3 + 3bQ_1^2Q_2 + 3cQ_1Q_2^2 + eQ_2^3 \tag{4.8}$$

where a is the probability that all three sites emerging from the origin are connected to it (figure 10(a)). Similar definitions are used for b, c and e (figures 10(b), (c) and (e)). Using the exclusion–inclusion principle, it is easy to obtain

$$\begin{aligned} a &= (r^3 + 3r^2s)(1 - q^3) + 3rs^2p(1 - q^2) + s^3p^3 \\ b &= [r(1 - q^2) + sp^2]qs^2 \\ c &= pq^2s^2 \\ e &= q^3. \end{aligned} \tag{4.9}$$

Again, the normalisation condition $a + 3b + 3c + e = 1$ holds for (4.9).

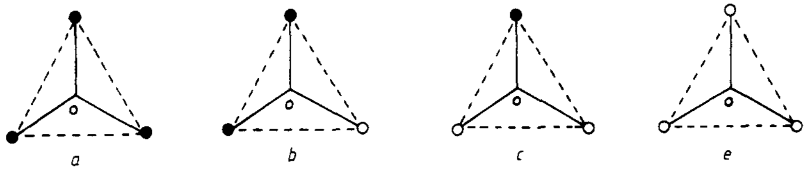

Figure 10. Various configurations for the probabilities a, b, c and e.

To find the mean cluster size, we use the 'ghost site' method described in § 2. The calculation is straightforward and will not be given here. The result of $S(p, r)$ for (p, r) below the critical line is given by

$$S(p, r)=1+3(a+2b+c)[1-(\Delta_1/\Delta)]-3(\Delta_2/\Delta)(b+2c+e) \tag{4.10}$$

where a, b, c and e are given by (4.9). Δ has the same expression as the left-hand side of (4.7) with all the subscripts 'c' omitted. Δ_1 and Δ_2 in (4.10) are given by

$$\begin{aligned}\Delta_1&=2-2qs^2(1-qs+q)+4(1-s^2-s^2r)[3qs^2(1-qs+q)-2(sqp+q^2)]\\ \Delta_2&=2(1-s^2-s^2r)[1+2qs^2(1-qs+q)-2(spq+q^2)].\end{aligned} \tag{4.11}$$

From (4.10), it is easy to see that $S(p, r)$ diverges on the whole critical line with the same exponent $\gamma=1$ as the site percolation case. We would expect this site–bond universality to hold for all $K>1$ and for any L. $(S(p, r))^{-1}$ is plotted numerically in figure 11.

The method described in this section can easily be generalised formally to the case of higher values of K and L. However, the calculations will be much more complicated and will not be pursued here.

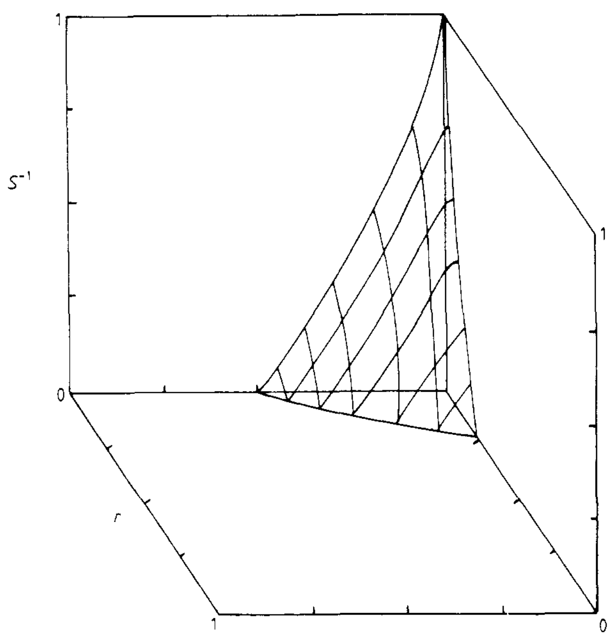

Figure 11. $(S(p, r))^{-1}$ for bond percolation with $K=2$ and $L=2$, with interbranch bonds considered.

5. Summary

The recursive method is generalised to treat percolation on a Bethe lattice with Lth-nearest-neighbour bonds. In the site percolation, both the cases without and with interbranch bonds are considered. Formal expressions for the critical percolation p_c, percolation probability $P(p)$ (near p_c) and the mean cluster size $S(p)$ ($p<p_c$) are obtained for any K and L. For $K>1$ and for any L, the critical behaviour of the models belongs to the same universality class. In the bond percolation, only the case of $K=2$ and $L=2$ is studied. Formal expressions for the critical line ($f(p_c, r_c)=0$) and the mean cluster sizes $S(p, r)$ (below the critical line) are obtained. The method described here can be generalised to treat other more complicated branching media including decorated Bethe lattices.

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Appendix

In this appendix, we will give only a brief account of how (2.7) is derived. Expanding (2.2) to second order in λ we find

$$\sum_{j=0}^{L-1} A_{ij} Q_j''(0) = C_i \qquad i=0,1,\ldots,L-1 \tag{A1}$$

where

$$C_i = -[(K-1)/K](Q_i'(0))^2 - 2KQ_0'(0) + 2KQ_{i+1}'(0) \tag{A2}$$

with

$$Q_L'(0)=0.$$

In (A1), A_{ij} is given by (2.4). The solvability condition for (A1) is the existence of the solution R_i ($i=0,1,\ldots,L-1$) of the equation (Stakgold 1979)

$$\sum_{j=0}^{L-1} A_{ji} R_j = 0 \qquad i=0,1,\ldots,L-1 \tag{A3}$$

with the condition that R_i must be orthogonal to C_i. That is

$$\sum_{i=0}^{L-1} R_i C_i = 0. \tag{A4}$$

In (A3), A_{ji} is the adjoint of A_{ij}. Using (2.4), (A3) has the form

$$R_i = K\left(p_c\delta_{0,i}\sum_{j=0}^{L-1} R_j + q_c R_{i-1}\right). \tag{A5}$$

The solution of (A5) is

$$R_i = (Kq_c)^i R_0 \qquad i=1,2,\ldots,L-1 \tag{A6}$$

with

$$R_0 = Kp_c\left(\sum_{j=0}^{L-1} R_j\right). \tag{A7}$$

Substituting (A6), (A7) and (A2) into (A4), with some straightforward manipulation, we finally arrive at (2.7).

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